Chapter 5 Linear Programming (LP)

- General constrained optimization problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in \Omega \end{array}$$

- $\Omega \subset R^n$ is called the constraint set or feasible set.

- any point $x \in \Omega$ is called a feasible point

- We consider the case when $f(x)$ is a linear function

$$f(x) = c^T x, \quad x = (x_1, x_2, ..., x_n) \in R^n$$

 where $c \in R^n$ is a given constant vector, called cost coefficient.

- Example: R^3

$$f(x) = 3x_1 - 4x_2 + x_3 = c^T x, \quad c = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

- Consider standard convex constraint set

$$\Omega = \{x : Ax = b, \ x \geq 0\}$$

- The notation $x \geq 0$ means that each component of $x = [x_1, ..., x_n]^T$ must be $\geq 0 : x_i \geq 0$ for all i

- **Example:** $\Omega = \{x : 4x_1 + x_2 = 5, \ x_1 \geq 0, x_2 \geq 0\}$
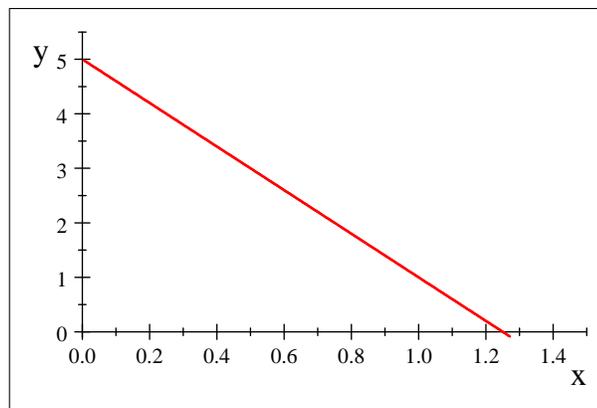
- We can write the problem as

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

- This problem is called Linear programming problem, or linear programming (LP).

- In general, constraint sets involve equations and inequalities.

- **Example:**

$$\begin{aligned} \text{minimize} \quad & 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 - x_2 \leq 6 \\ & x_1 \leq 0 \end{aligned}$$

- **Example** Production scheduling

- Woodworking shop data:

Inputs	Products		input availability
	Table	Chair	
labor	5	6	30
Materials	3	2	12
Production Levels	x_1	x_2	
Unit price	1	5	

- Goal: schedule production to maximize revenue

- Total revenue: $x_1 + 5x_2$

- Production constraints

    - Labor constraint: $5x_1 + 6x_2 \leq 30$
    - Materials constraint: $3x_1 + 2x_2 \leq 12$
    - Physical constraints: $x_1 \geq 0, x_2 \geq 0$

- Optimization problem:

$$
\begin{aligned}
\text{maximize} \quad & x_1 + 5x_2 \\
\text{subject to} \quad & 5x_1 + 6x_2 \leq 30 \\
& 3x_1 + 2x_2 \leq 12 \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

- (LP) in matrix form

$$
\begin{aligned}
\text{maximize} \quad & c^T x \\
\text{subject to} \quad & Ax \leq b \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

$$
A = \begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 30 \\ 12 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 5 \end{bmatrix}
$$

- Equivalent form

$$
\begin{aligned}
\text{minimize} \quad & -c^T x \\
\text{subject to} \quad & Ax \leq b \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

Example: Optimal Diet

- Nutrition table:

Unit Price	Vitamin	Milk	Eggs	Daily Requirement
λ_1	F	3	7	45
λ_2	G	4	2	60
	Intake	x_1	x_2	
	Unit cost	2	5	

- For consumers:

  - Total cost: $2x_1 + 5x_2$
  - Dietary Constraints
    * Vitamin F: $3x_1 + 7x_2 \geq 45$

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* Vitamin G: $4x_1 + 2x_2 \geq 60$
* Physical constraints: $x_1 \geq 0, x_2 \geq 0$

– Optimization problem:

$$
\begin{aligned}
\text{minimize} \quad & 2x_1 + 5x_2 \\
\text{subject to} \quad & 3x_1 + 7x_2 \geq 45 \\
& 4x_1 + 2x_2 \geq 60 \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

– (LP) in matrix form

$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Ax \geq b \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

$$
A = \begin{bmatrix} 3 & 7 \\ 4 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 45 \\ 60 \end{bmatrix}, \quad c = \begin{bmatrix} 2 \\ 5 \end{bmatrix}
$$

– This is called Primal LP

• For sellers:

– total revenue (per customer) $45\lambda_1 + 60\lambda_2$
– to compete well with milk: $3\lambda_1 + 4\lambda_2 \leq 2$
– to compete well with egg: $7\lambda_1 + 2\lambda_2 \leq 5$
– LP for seller:

$$
\begin{aligned}
\text{Maximize} \quad & 45\lambda_1 + 60\lambda_2 \\
\text{Subject to} \quad & 3\lambda_1 + 4\lambda_2 \leq 2 \\
& 7\lambda_1 + 2\lambda_2 \leq 5 \\
& \lambda_1, \lambda_2 \geq 0
\end{aligned}
$$

– This is called dual LP.

Geometric method for LP problems in 2-D

- Consider the production scheduling example given before:

$$\begin{aligned} \text{maximize} \quad & x_1 + 5x_2 \\ \text{subject to} \quad & 5x_1 + 6x_2 \leq 30 \\ & 3x_1 + 2x_2 \leq 12 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$
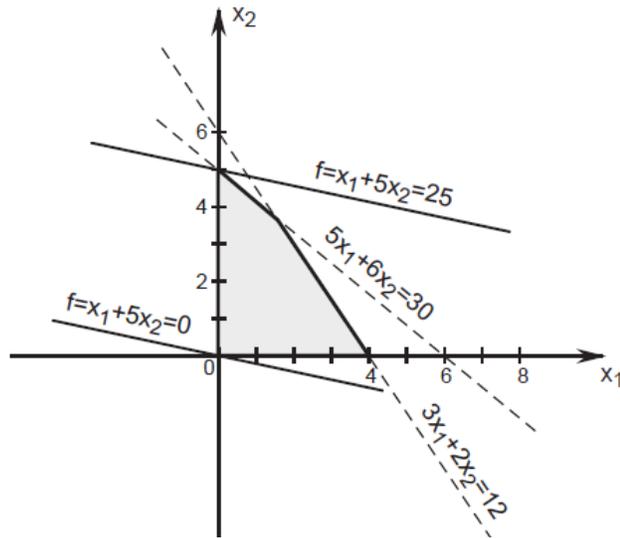
- Level sets of the objective function: $x_1 + 5x_2 = c$

- Level sets are parallel lines, moving up as c increases

- The maximum value 25 is reached when the level set "exiting" the constraint set at the corner point $[0,5]^T : x_1 + 5x_2 = 25$

- The minimum value 0 is reached when the level set "exiting" the constraint set at the corner point $[0,0]^T : x_1 + 5x_2 = 0$

- In fact, unless the level sets happen to be parallel to one of the edges of the constraint set, the solution will lie on a "corner point" (or vertex). Even if the level sets happen to be parallel to one of the edges of the constraint set, a corner point will be an optimal solution.

- It turns out that solution of an LP problem (if it exists) always lies on a vertex of the constraint set.

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- Therefore, instead of looking for candidate solutions everywhere in the constraint set, we need only focus on the vertices.

Standard form LP problems

- LP problem in standard form:

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
& x \geq 0
\end{array}
$$

where A is a $m \times n$ matrix, $m < n$,

$$
\begin{array}{c}
rank\,(A) = m \\
b \geq 0
\end{array}
$$

- Example:

    - In standard form:

    $$
    \begin{array}{ll}
    \text{minimize} & 3x_1 + 5x_2 - x_3 \\
    \text{subject to} & x_1 + 2x_2 + 4x_3 = 4 \\
    & -5x_1 - 3x_2 + x_3 = 15 \\
    & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
    \end{array}
    $$

    - NOT in standard forms:

    $$
    \begin{array}{ll}
    \text{maximize} & 3x_1 + 5x_2 - x_3 \\
    \text{subject to} & x_1 + 2x_2 + 4x_3 \leq 4 \\
    & -5x_1 - 3x_2 + x_3 \geq 15 \\
    & x_1 \geq 0, x_2 \geq 0
    \end{array}
    $$

    $$
    \begin{array}{ll}
    \text{minimize} & 3x_1 + 5x_2 - x_3 \\
    \text{subject to} & x_1 + 2x_2 + 4x_3 = -4 \\
    & -5x_1 - 3x_2 + x_3 = 15 \\
    & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
    \end{array}
    $$

– The above is not standard because $b_1 = -4 < 0$. To be standard form, we require $b_i \geq 0$.

$$
\begin{array}{ll}
\text{minimize} & 3x_1 + 5x_2 - x_3 \\
\text{subject to} & x_1 + 2x_2 + 4x_3 = 4 \\
& 2x_1 + 4x_2 + 8x_3 = 8 \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{array}
$$

– The above is not standard because the matrix $A_{2\times 4}$ has a rank of 1, not "full rank".

- All our analyses and algorithms will apply only to standard form LP problems.

- What about other variations of LP problems?

  – Any LP problem can be converted into an equivalent standard form LP problem.

- How to convert from given LP problem to an equivalent problem in standard form?

  – If problem is a maximization, simply multiply the objective function by -1 to get minimization.

  – If A not of full rank, can remove one or more rows.

  – If a component of b is negative, say the ith component, multiply the ith constraint by -1 to obtain a positive right-hand side.

- What about inequality constraints?

  – add slack variables for " $\leq$ " constraints

  – add surplus variables for " $\geq$ " constraint

  – split variables for free variables

Converting to standard form: Slack variables

- Suppose we have a " $\leq$ " constraint of the form:

$$
a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b
$$

- We can convert the above inequality constraint into the standard equality constraint by introducing a slack variable x_{n+1}

- The above constraint is equivalent to:

$$a_1 x_1 + a_2 x_2 + \cdots a_n x_n + x_{n+1} = b$$
$$x_{n+1} \geq 0$$

Converting to standard form: Surplus variables

- Suppose we have a constraint of the form:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq b$$

- We can convert the above inequality constraint into the standard equality constraint by introducing a surplus variable x_{n+1}

- The above constraint is equivalent to:

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n - x_{n+1} = b$$
$$x_{n+1} \geq 0$$

Converting to standard form: Nonpositive variable

- Suppose one of the variables (say, x_1) has the constraint

$$x_1 \leq 0$$

- We can convert the variable into the usual nonnegative variable by changing every occurrence of x_1 by its negative $x_1' = -x_1$

- Example: The LP problem

$$\begin{aligned}
\text{minimize} \quad & c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \\
\text{subject to} \quad & a_1 x_1 + a_2 x_2 + \cdots + a_n x_n - a_{n+1} x_{n+1} = b \\
& x_1 \leq 0
\end{aligned}$$

is converted standard form

$$\begin{aligned}
\text{minimize} \quad & -c_1 x_1' + c_2 x_2 + \cdots + c_n x_n \\
\text{subject to} \quad & -a_1 x_1' + a_2 x_2 + \cdots + a_n x_n - a_{n+1} x_{n+1} = b \\
& x_1' \geq 0
\end{aligned}$$

Converting to standard form: Free variables

- Suppose one of the variables (say, x_1) does not have any constraint

- We can split x_1 into its positive part and negative part by introducing two variables $u_1 \geq 0$ and $v_1 \geq 0$: $x_1 = u_1 - v_1$

- Example: The constraint

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$
$$x_2 \geq 0, x_3 \geq 0, \cdots, x_n \geq 0$$

  is equivalent to

$$a_1 (u_1 - v_1) + a_2 x_2 + \cdots + a_n x_n = b$$
$$u_1 \geq 0, v_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \cdots, x_n \geq 0$$

Converting to standard form: An example

- Consider the LP problem (not in standard form)

$$
\begin{aligned}
\text{maximize} \quad & 3x_1 + 5x_2 - x_3 \\
\text{subject to} \quad & x_1 + 2x_2 + 4x_3 \leq -4 & (1)\\
& -5x_1 - 3x_2 + x_3 \geq 15 & (2)\\
& x_2 \leq 0 \\
& x_3 \geq 0
\end{aligned}
$$

- We first convert the maximization problem into minimization problem by multiplying the objective function by (-1)

- Next, we introduce a slack variable x_4 in the first constraint to convert constraint (1) into

$$x_1 + 2x_2 + 4x_3 + x_4 = -4, \quad x_4 \geq 0$$

  and then multiply the equation by (-1) to make the right-hand side positive:

$$-x_1 - 2x_2 - 4x_3 - x_4 = 4, \quad x_4 \geq 0$$

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- Next, we introduce a surplus variable x_5 to convert the second constraint (2) to convert it to

$$-5x_1 - 3x_2 + x_3 - x_5 = 15, \ x_5 \geq 0$$

- Next, we substitute x_1 with $x_6 - x_7$ in every expression

- Finally, we substitute x_2 with $-x_2'$ in every expression

- The resulting standard form

$$
\begin{aligned}
\text{minimize} \quad & -3(x_6 - x_7) + 5x_2' + x_3 \\
\text{subject to} \quad & -(x_6 - x_7) + 2x_2' - 4x_3 - x_4 = 4 \\
& -5(x_6 - x_7) + 3x_2' + x_3 - x_5 = 15 \\
& x_2', x_3, x_4, x_5, x_6, x_7 \geq 0
\end{aligned}
$$

or

$$
\begin{aligned}
\text{minimize} \quad & 5x_2' + x_3 - 3x_6 + 3x_7 \\
\text{subject to} \quad & 2x_2' - 4x_3 - x_4 + 0x_5 - x_6 + x_7 = 4 \\
& 3x_2' + x_3 + 0x_4 - x_5 - 5x_6 + 5x_7 = 15 \\
& x_2', x_3, x_4, x_5, x_6, x_7 \geq 0
\end{aligned}
$$

The matrix form is

$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Ax = b \quad, \quad x = \begin{bmatrix} x_2' & x_3 & x_4 & x_5 & x_6 & x_7 \end{bmatrix}^T \\
& x \geq 0
\end{aligned}
$$

$$
A = \begin{bmatrix} 2 & -4 & -1 & 0 & -1 & 1 \\ 3 & 1 & 0 & -1 & -5 & 5 \end{bmatrix}, \ b = \begin{bmatrix} 4 \\ 15 \end{bmatrix}
$$

Basic solutions

- Consider LP problem in standard form:

$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Ax = b \\
& x \geq 0
\end{aligned}
$$

where A is a $m \times n$ matrix, $m < n, \ rank(A) = m, \ b \geq 0$

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- The **feasible points** are those x that satisfy $Ax = b$ and $x \geq 0$ (i.e., have nonnegative components).

- For $m < n$, there are infinitely many points x satisfying $Ax = b$

REVIEW linear algebra:

- Recall there are three types of elementary row operations

  1. Interchanging two rows of the matrix;
  2. Multiplying one row of the matrix by a (nonzero) constant;
  3. Adding a constant multiple of one row to another row.

- Pivoting about (i,j) position: using above three types of row operations to make *the* entry at (i,j) position 1, and the rest entries in the jth column zero

- To solve $Ax = b$ using elementary row operations:

  – First, we form the "augmented matrix" $[A, b]$
  – Using Gauss-Jordan Algorithm
  – $[A, b]$ can be reduced by a sequence of elementary row operations to a unique reduced Echelon form.
  – Assume that the first m columns of A are linearly independent
  – **pivoting** about $(1, 1), (2, 2), ..., (m, m)$ position consecutively
  – this will reduce $[A, b]$ to the reduced Echelon

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & y_1 \\ 0 & 1 & \cdots & 0 & \cdots & y_2 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & \cdots & y_m \end{bmatrix} = [I_m, Y, y^*], \quad y^* = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

  – Name: canonical augmented matrix

- In particular, $Ax = b$ has a solution

$$x^* = \begin{bmatrix} y^* \\ 0 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

  where y^* is the last column in the above canonical augmented matrix

- **Basic solution:** This solution $x^* = \begin{bmatrix} y^{*T}, 0 \end{bmatrix}^T$ is called a basic solution to $Ax = b$

- The first m columns of A are called **basic columns**

- The first m variables $x_1, \cdots, x_m$ are called **basic variables**

- The last $(n - m)$ variables $x_{m+1}, \cdots, x_n$ are called **nonbasic variables**

- In general, for any solution x^* of $Ax = b$

  - x^* is called a **basic solution** if x^* has at most m non-zero components

  - the corresponding variables (of non-zero components) are called **basic variables**

  - the corresponding columns in A are called **basic columns** that form a basis

- For any m basic columns of A : $a_{k_1}, a_{k_2}, ..., a_{k_m}$ are linearly independent, where $1 \leq k_1 \leq k_2 \leq ... \leq k_m$.

- Canonical augmented matrix: k_1-th column is $e_1 = [1, 0, ..., 0]^T$, k_2-th column is $e_2, .., k_m$-th column is e_m

$$[A, b] = [a_1, a_2, ..., a_n, b] \rightarrow \begin{bmatrix} \cdots & 1 & \cdots & 0 & \cdots & 0 & \cdots \\ \cdots & 0 & \cdots & 1 & \cdots & 0 & \cdots \\ \cdots & 0 & \vdots & 0 & \cdots & 1 & \cdots \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots \end{bmatrix}$$

- In this case, $a_{k_1}, a_{k_2}, ..., a_{k_m}$ are basic columns, variables associated with basic columns, $x_{k_1}, x_{k_2}, ..., x_{k_m}$, are basic variables.

- Example:

$$A_{2\times 4} = [a_1, a_2, a_3, a_4] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

  - canonical augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 6 \\ 5 & 6 & 7 & 8 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{3}{2} & 1 & \frac{1}{2} & 0 & -5 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 1 & 4 \end{bmatrix}$$

  - x_2 and x_4 are basic variables
  - a_2, a_4 are basic columns
  - a solution x^*

$$x^* = \begin{bmatrix} 0 \\ -5 \\ 0 \\ 4 \end{bmatrix}$$

  - this $x^* = [0, -5, 0, 4]^T$ is a basic solution ($m = 2$)

- There are at most

$$\binom{n}{m} = \frac{n!}{m! \, (n - m)!} \quad \text{basic solutions}$$

Example Consider the equation $Ax = b$ with

$$A = [a_1, a_2, a_3, a_4] = \begin{bmatrix} 1 & 1 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 4 & 8 \\ 1 & -2 & -1 & 1 & 2 \end{bmatrix}$$

- #1 basic solution: $x = [6, 2, 0, 0]^T$, basic variables x_1, x_2, basis $B = [a_1, a_2]$

- #2 basic solution: $x = [0, 2, -6, 0]^T$, basic variables x_2, x_3, basis $B = [a_2, a_3]$

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- #3 basic solution: $x = [0, 0, 0, 2]^T$, basic variables x_3, x_4, basis $B = [a_3, a_4]$

- #4 basic solution: $x = [0, 0, 0, 2]^T$, basic variables x_1, x_4, basis $B = [a_1, a_4]$

- #5 basic solution: $x = [0, 0, 0, 2]^T$, basic variables x_2, x_4, basis $B = [a_2, a_4]$

- Note that $x = [3, 1, 0, 1]^T$ is a solution, but it is not basic

- To find the last basic solution, we set $x_2 = x_4 = 0$

- the solution of form $x = [x_1, 0, x_3, 0]^T$ for $Ax = b$ is equivalent to

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

- its augmented matrix

$$\begin{bmatrix} 1 & -1 & 8 \\ 1 & -1 & 2 \end{bmatrix}$$

- is inconsistent. So there are only five basic solutions

Basic Feasible Solutions (BFS)

- Consider the constraints of the LP problem: $Ax = b, x \geq 0$

- Recall definition of basic solution

- If x is a basic solution to $Ax = b$ and it also satisfies $x \geq 0$, we call it a basic feasible solution (BFS)

- Note that a basic solution x is feasible iff every basic variable is ≥ 0.

- If at least one of the basic variables is $= 0$, we say that the BFS is degenerate

- We henceforth assume nondegeneracy (i.e., all the BFSs we consider are not degenerate)

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- This makes the derivations simpler. The degenerate case can be handled similarly.

- In Example above:

  - $x = [6, 2, 0, 0]^T$ is a BFS
  - $x = [0, 0, 0, 2]^T$ is a degenerate BFS
  - $x = [0, 2, -6, 0]^T$ is a basic solution, but not feasible
  - $x = [3, 1, 0, 1]^T$ is feasible, but not basic

Geometric view

- Geometrically, a BFS corresponds to a "corner" point (vertex) of the constraint set

- **Example:** Consider the constraint $Ax = b, x \geq 0$, where

$$A = \begin{bmatrix} 1, & 2, & 3 \end{bmatrix}, \ b = [6].$$

- Feasible set the portion of the plane $x_1 + 2x_2 + 3x_3 = 6$ in the first octant

- The BFSs are: $[6, 0, 0]^T, [0, 3, 0]^T$, and $[0, 0, 2]^T$

- Note that the BFSs are just the vertices of the constraint set.

- In general, we can think (geometrically) of a BFS as a vertex of the constraint set.

Optimal solutions

- A feasible point that minimizes the objective function is called an optimal feasible solution

- A BFS that is also optimal is called an optimal basic feasible solution
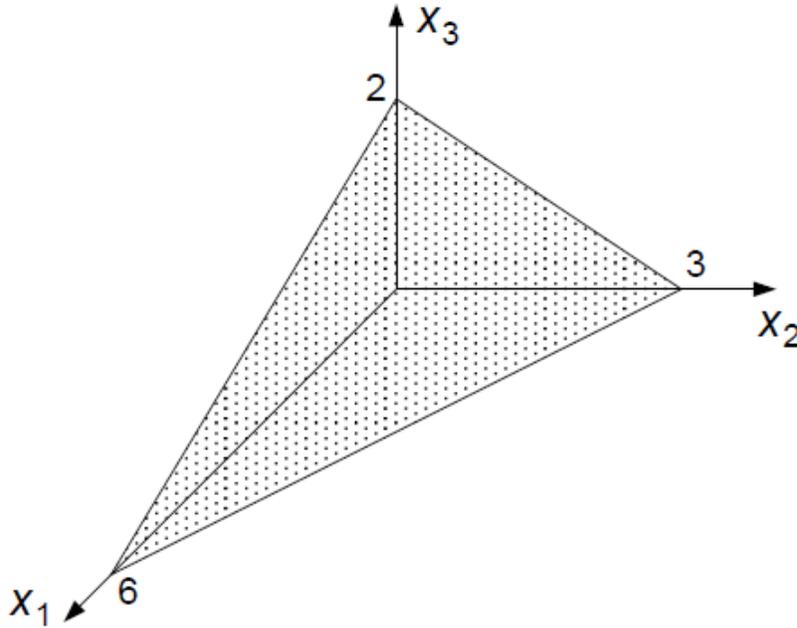
Fundamental Theorem of LP

- Consider an LP problem in standard form.

  1. If there exists a feasible solution, then there exists a BFS;
  2. If there exists an optimal feasible solution, then there exists an optimal BFS.

Or, more informally, ...

- If the feasible set is nonempty, then it has a vertex

- If the problem has a minimizer (optimal solution), then one of the vertices is a minimizer

Consequences of FTLP

- Suppose we are given an LP problem. Assume that a solution exists.

- To find the solution, it suffices to look among the set of BFSs.

- There are only a finite number of BFSs; in fact, at most

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!} \quad \text{basic solutions}$$

- Therefore, the FTLP allows us to transform the original problem (infinite number of feasible points) to a problem over a finite number of points

- However, the total number of BFSs may of course be very large.

- For example, if $m = 5$ and $n = 50$, $\binom{50}{5} = 2\,118\,760$

- The brute force approach of exhaustively comparing all the BFSs is impractical

- Therefore, we need a more computationally efficient method to find the minimizer among the possible BFSs

- Simplex method: an organized way of going from one BFS to another to search for the global minimizer.

Simplex Algorithms

- The FTLP allows us to limit search of optimal solutions to a finite number of points (BFSs).

- Simplex method: an organized way of going from one BFS to another to search for the global minimizer.

- The method uses numerical linear algebraic techniques.

- Sometimes, to avoid confusion, we use boldface letters for vectors

Simplex method
Basic idea

- Start with an initial basis corresponding to a BFS

- We then move to an adjacent BFS (adjacent vertex) in such a way that the objective function decreases

- If the stopping criterion is satisfied, we stop; otherwise, we repeat the process.

- We move from one basic solution to an adjacent one:

  - if $\mathbf{a}_p$ leaves the basis and $\mathbf{a}_q$ enters the basis, we pivot about the (r, q) position of the canonical augmented matrix

$$
\begin{bmatrix}
\cdots & \cdots & p\text{-th column} & \cdots & \cdots \\
\cdots & \cdots & \vdots & \cdots & \cdots \\
\cdots & \cdots & 0 & \cdots & \cdots \\
r\text{-th row} & \cdots & 1 & \cdots & \cdots \\
\cdots & \cdots & 0 & \cdots & \cdots \\
\cdots & \cdots & \vdots & \cdots & \cdots \\
\cdots & \cdots & 0 & \cdots & \cdots
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\cdots & \cdots & \cdots & q - \text{th column} & \cdots \\
\cdots & \cdots & \cdots & \vdots & \cdots \\
\cdots & \cdots & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & 1 & \cdots \\
\cdots & \cdots & \cdots & 0 & \cdots \\
\cdots & \cdots & \cdots & \vdots & \cdots \\
\cdots & \cdots & \cdots & 0 & \cdots
\end{bmatrix}
$$

  - x_p is D.V.(departing variable)

  - x_q is E.V. (entering variable)

  - Example: $\mathbf{a}_2, \mathbf{a}_4$ form the original basis. To replace $\mathbf{a}_2$ by $\mathbf{a}_1$, we need to pivot about $(1, 1)$ $(p = 2, r = 1, q = 1)$. The new basis is $\{\mathbf{a}_1, \mathbf{a}_4\}$

$$
\begin{bmatrix}
1^* & 1 & 3 & 0 & 8 \\
-2 & 0 & -1 & 1 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 3 & 0 & 8 \\
0 & 2 & 51 & 1 & 18
\end{bmatrix}
$$

  x_2 is D.V., x_1 is E.V.

- Two questions:

  1. How do we ensure that the adjacent basic solution we move to is feasible?

     - In other words if we are given x_q E.V., how do we choose a D.V. x_p such that if we pivot about the (r, q) element $((r, p)$ is a pivot), the resulting rightmost column of the canonical augmented matrix will have all nonnegative elements?

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2. How do we choose E.V. x_q to ensue the objective function at the new basic solution is smaller the previous value?

Maintain feasibility (optional): Choosing leaving variable x_p

- Feasibility: the last column ≥ 0

- Given: A and b.

- Suppose we have a set of basic variables corresponding to a BFS.

- For simplicity, assume it is $\{x_1, ..., x_m\}$, the corresponding basis is $\{\mathbf{a}_1, ..., \mathbf{a}_m\}$

- Suppose we now want to choose x_q as E.V., where $q > m$.

- One of the vectors $\mathbf{a}_p$, $1 \leq p \leq m$, must leave the basis.

- Suppose the canonical augmented matrix for the original basis is:

$$[A, b] \rightarrow \begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1,m+1} & \cdots & y_{1,n} & y_{1,0} \\ 0 & 1 & \cdots & 0 & y_{2,m+1} & \cdots & y_{2,n} & y_{2,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{m,m+1} & \cdots & y_{m,n} & y_{m,0} \end{bmatrix}$$

- Note $y_{i0} \geq 0$ due to feasible assumption

- If, for example, $\mathbf{a}_{m+1}$ replaces $\mathbf{a}_1$ by pivoting bout $(1, m+1)$

- the resulting canonical augmented matrix

$$[A, b] \rightarrow \begin{bmatrix} \frac{1}{y_{1,m+1}} & 0 & \cdots & 0 & 1 & \cdots & \frac{y_{1,n}}{y_{1,m+1}} & \frac{y_{1,0}}{y_{1,m+1}} \\ 0 & 1 & \cdots & 0 & y_{2,m+1} & \cdots & y_{2,n} & y_{2,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{m,m+1} & \cdots & y_{m,n} & y_{m,0} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \frac{1}{y_{1,m+1}} & 0 & \cdots & 0 & 1 & \cdots & \frac{y_{1,n}}{y_{1,m+1}} & & \frac{y_{1,0}}{y_{1,m+1}} \\ -\frac{y_{2,m+1}}{y_{1,m+1}} & 1 & \cdots & 0 & 0 & \cdots & y_{2,n} - \frac{y_{1,n}}{y_{1,m+1}}y_{2,m+1} & y_{2,0} - \frac{y_{1,0}}{y_{1,m+1}}y_{2,m+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ \frac{y_{m,m+1}}{y_{1,m+1}} & 0 & \cdots & 1 & 0 & \cdots & y_{m,n} - \frac{y_{1,n}}{y_{1,m+1}}y_{m,m+1} & y_{m,0} - \frac{y_{1,0}}{y_{1,m+1}}y_{m,m+1} \end{bmatrix}$$

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- the last column is

$$
\begin{bmatrix}
\dfrac{y_{10}}{y_{1m+1}} \\[4pt]
y_{20} - \dfrac{y_{10}}{y_{1m+1}} y_{2m+1} \\[4pt]
\vdots \\[4pt]
y_{m0} - \dfrac{y_{10}}{y_{1m+1}} y_{mm+1}
\end{bmatrix}
$$

- In general, if $\mathbf{a}_q$ replaces $\mathbf{a}_p$ by pivoting about (r,q) $((r,p)$ is a pivot), the last column of new canonical augmented matrix will be

$$
\begin{bmatrix}
\vdots \\[4pt]
\dfrac{y_{r,0}}{y_{r,q}} & & r\text{-th} \\[6pt]
\vdots \\[4pt]
y_{i,0} - \dfrac{y_{r,0}}{y_{r,q}} y_{i,q} & & i\text{-th} \\[6pt]
\vdots
\end{bmatrix}
$$

- i.e.,

  - $r-$th element is $\dfrac{y_{r,0}}{y_{r,q}}$

  - all other $i-$th element is $y_{i,0} - \dfrac{y_{r,0}}{y_{r,q}} y_{i,q}$ for $i \neq p$

- If $y_{r,q} < 0$, then the feasible set is unbounded

- Hence, to ensue feasibility, we have to choose r such that $y_{r,q} > 0$, and for all $i = 1, 2, ..., m$,

$$
y_{i0} - \frac{y_{r,0}}{y_{r,q}} y_{i,q} = y_{i,q}\left(\frac{y_{i,0}}{y_{i,q}} - \frac{y_{r,0}}{y_{r,q}} \right) \geq 0 \implies
$$

$$
r = \arg\min \left\{ \frac{y_{i,0}}{y_{i,q}} : i = 1, 2, ..., m \right\}
$$

- We may divide i-th component in q-th column into i-th component in the last column to create "division"

$$
\begin{bmatrix}
y_{1,0} \\
y_{2,0} \\
\vdots \\
y_{m,0}
\end{bmatrix}
\div
\begin{bmatrix}
y_{1,q} \\
y_{2,q} \\
\vdots \\
y_{m,q}
\end{bmatrix}
=
\begin{bmatrix}
y_{1,0}/y_{1,q} \\
y_{2,0}/y_{2,q} \\
\vdots \\
y_{m,0}/y_{m,q}
\end{bmatrix}
$$

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- In summary, to maintain feasibility, the leaving variable x_p must be the position in the above "quotation" vector with **the smallest value**, i.e.,

$$\frac{y_{r,0}}{y_{r,q}} \le \frac{y_{i,0}}{y_{i,q}}$$

Maintain optimality (optional): Choosing entering variable x_q

- Finding a "better" adjacent BFS

- Once we select $\mathbf{a}_q$, we know how to pick $\mathbf{a}_p$ to leave the basis

- The question is how to choose column $\mathbf{a}_q$ to enter the basis?

- The goal is that the new (adjacent) BFS should have lower objective function value

- Suppose the canonical augmented matrix for the original basis is:

$$[A, b] = \begin{bmatrix} 1 & 0 & \cdots & 0 & y_{1,m+1} & \cdots & y_{1,n} & y_{1,0} \\ 0 & 1 & \cdots & 0 & y_{2,m+1} & \cdots & y_{2,n} & y_{2,0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & y_{m,m+1} & \cdots & y_{m,n} & y_{m,0} \end{bmatrix}$$

- basic cost function coefficients are those associated with the basic variables

$$\mathbf{c}_0 = [c_1, c_2, \cdots, c_m]$$

- The objective function for the BFS $x^* = [y_{1_0}, \cdots, y_{m0}, 0, \cdots 0]^T$ is

$$z_0 = c^T x^* = \mathbf{c}_0^T x^* = \sum_{i=1}^{m} c_i y_{i0}$$

- Now, consider a new basis where $\mathbf{a}_q$ enters the basis and $\mathbf{a}_p$ leaves the basis.

- The new BFS is (with $r = p$ in this case)

$$x^{**} = \left[y_{1,0} - \frac{y_{r,0}}{y_{r,q}} y_{1,q}, \cdots 0, \cdots, y_{m,0} - \frac{y_{r,0}}{y_{r,q}} y_{m,q}, 0, \cdots 0, \frac{y_{r,0}}{y_{r,q}}, 0 \cdots \right]$$

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- The objective function for the new BFS is

$$z = c^T x^{**} = \sum_{i=1, i \neq r}^{m} c_i \left(y_{i0} - \frac{y_{r0}}{y_{rq}} y_{iq} \right) + c_q \frac{y_{r0}}{y_{rq}}$$

$$= \sum_{i=1}^{m} c_i \left(y_{i0} - \frac{y_{r0}}{y_{rq}} y_{iq} \right) + c_q \frac{y_{r0}}{y_{rq}}$$

$$= \sum_{i=1}^{m} c_i y_{i0} - \sum_{i=1}^{m} c_i \frac{y_{r0}}{y_{rq}} y_{iq} + c_q \frac{y_{r0}}{y_{rq}}$$

$$= z_0 + \left(c_q - \sum_{i=1}^{m} c_i y_{i,q} \right) \frac{y_{r,0}}{y_{r,q}}$$

- Note that the q-th column of A is

$$\begin{bmatrix} y_{1,q} \\ y_{2,q} \\ \vdots \\ y_{m,q} \end{bmatrix}$$

- Hence q-th component in $\mathbf{c}_0^T A$ is

$$z_q = \sum_{i=1}^{m} c_i y_{i,q}$$

- we have
$$z = z_0 + (c_q - z_q) \frac{y_{r,0}}{y_{r,q}}$$

- Hence if $(c_q - z_q) < 0$, we conclude $z < z_0$, i.e., the objective function for the new BFS is smaller

- Define

$$r_i = \begin{cases} 0 & \text{for } i = 1, ..., m \ \ (\text{basic}) \\ c_i - z_i & \text{for } i = m + 1, ..., n \ \ (\text{nonbasic}) \end{cases}$$

- we call r_i Relative cost coefficients (RCC) for x_i

$$z = z_0 + r_q \frac{y_{r,0}}{y_{r,q}}$$

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- Note: the RCC for a basic variable is always 0.

- If $r_q < 0$, then the new BFS is better.

- The smaller r_q, the better the new BFS

- In general, if x is any feasible solution, we can derive the equation

$$z = c^T x = z_0 + \sum_{i=m+1}^{n} r_i x_i$$

- RCC (Relative cost coefficients) vector $\mathbf{r}$ is

$$\mathbf{r} = [r_0, r_2, ..., r_n]^T = c^T - c_0^T A$$

- **Optimality conditions**: $\mathbf{r} = c^T - c_0^T A \geq 0$.

- This leads to

Theorem A BFS is optimal if and only if the corresponding RCC values (components of $c^T - c_0^T A$) are all ≥ 0.

- In summary, we choose a variable x_q that has the most negative RCC value as the entering variable.

- The Simplex Algorithm

1. Initialization: Given an initial BFS and its canonical augmented matrix.

2. Calculate the RCCs ($c^T - c_0^T A$) corresponding to the nonbasic variables.

3. If $r_j > 0$ all j, then STOP—the current BFS is optimal.

4. Otherwise, select a q such that $r_q < 0$ is the most negative.

5. If in the qth column a_q, no element $y_{iq} > 0$, then STOP—the problem is unbounded; else, calculate $r = argmin\{y_{i0}/y_{iq} : y_{iq} > 0\}$.

6. Update canonical augmented matrix by pivoting about the (r, q) element.

7. Go to step 2.

Simplex tableau

- We design a specific algorithm to compute optimal solutions

- We focus on the standard form:

$$\text{minimize:} \quad z = c^T x$$
$$\text{subject to:} \quad Ax = b, \ x \geq 0,$$

$$rank\,(A) = m, \ b > 0$$

- Simplex tableau

    - Suppose that we already know one set of basic variables x_0. Let c_0 be the basic cost coefficient in c associated with basic variable x_0. For instance, if $x_0 = (x_2, x_5, x_8)$ be the set of basic variables, then $c_0^T = [c_2, c_5, c_8]$.

    - **simplex tableau**

		x^T c^T	
x_0	c_0	A	b
		$\mathbf{r} = c^T - c_0^T A$	$-c_0^T b$

    where the second column from the left (c_0) and the second row from the top (c^T) are for convenience in calculating the last row, and can be deleted from the final tableau:

	x^T	
x_0	A	b
	$c^T - c_0^T A$	$-c_0^T b$

Example: Consider

$$\begin{aligned} \text{minimize:} \quad & f(x_1, x_2) = -120x_1 - 80x_2 \\ \text{subject to:} \quad & 2x_1 + x_2 \le 6 \\ & 7x_1 + 8x_2 \le 28. \end{aligned}$$

- with the standard form

$$\begin{aligned} \text{minimize:} \quad & s = -120\,(y_1 - z_1) - 80\,(y_2 - z_2) \\ \text{subject to:} \quad & 2\,(y_1 - z_1) + (y_2 - z_2) + w_1 = 6 \\ & 7\,(y_1 - z_1) + 8\,(y_2 - z_2) + w_2 = 28 \\ & y_1, y_2, z_1, z_2, w_1, w_2 \ge 0 \end{aligned}$$

- the initial feasible solution

$$x^* = \begin{bmatrix} y_1 \\ z_1 \\ y_2 \\ z_2 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 28 \end{bmatrix}.$$

- Thus, basic variables and basic cost coefficients:

$$x_0 = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad c_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

-
$$c^T - c_0^T A = c^T$$

- simplex tableau is

		y_1	z_1	y_2	z_2	w_1	w_2	
		-120	120	-80	80	0	0	
w_1	0	2	-2	1	-1	1	0	6
w_2	0	7	-7	8	-8	0	1	28
		-120	120	-80	80	0	0	0

- the final simplex tableau is

	y_1	z_1	y_2	z_2	w_1	w_2	
w_1	2	-2	1	-1	1	0	6
w_2	7	-7	8	-8	0	1	28
	-120^*	120	-80	80	0	0	0

Simplex method

- The simplex algorithm moves the initial feasible but non-optimal solution to an optimal solution while maintaining feasibility through the iterative procedure.

- The goal is to obtain optimality condition, i.e., the last row (except for the very last element) are all non-negative.

step 1. Locate the most negative number in the bottom row of the simplex tableau, excluding the last column. The variable associated with this column is E.V. We also call it the **work column**. If more than one candidate for the most negative numbers exist, pick any one.

step 2. Form ratios by dividing each **positive number** in the working column, excluding the last row, into the element in the same row and last column. Designate the element in the work column that yields the **smallest ration** as the **pivot element**. The basic variable associated with this pivot element is D.V. If no element in the work column is positive, stop. The problem has no solution.

step 3. Pivoting about the pivot element: Use elementary row operations to convert the pivot element to 1 and then to reduce all *other elements* in the work column to 0.

Step 4. Swap E.V. with D.V. Replace the variable in the pivot row and the first column by the variable in the first row and work column. The new first column is the current set of basic variables.

Step 5. Repeat Step 1 through Step 4 until there are no negative numbers in the last row, excluding the last column. The basic feasible solution derived from the very last set of variable is a minimal point: assign to each basic variable (variables in the first column) the number in

the same row and last column, and all others are zero. The **negative** number of the number in the last row and last column is the minimum value.

Example: Consider the example above with the standard form copied below:

	y_1	z_1	y_2	z_2	w_1	w_2	
w_1	2*	-2	1	-1	1	0	6
w_2	7	-7	8	-8	0	1	28
	-120	120	-80	80	0	0	0

Solution:

- Step 1. The work column is the first in the middle (-120 is the most negative).

- Step 2. The ratios are

$$\begin{bmatrix} 6 \\ 28 \end{bmatrix} \div \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

  So the element 2* is the pivot. w_1 is D.V. and y_1 is E.V.

- Step 3. We perform row operations to reduce to the following updated simplex tableau:

	y_1	z_1	y_2	z_2	w_1	w_2	
y_1	1	-1	$1/2$	$-1/2$	$1/2$	0	3
w_2	0	0	$9/2$**	$-9/2$	$-7/2$	1	7
	0	0	-20*	20	60	0	360

  The basic variable w_1 is replaced by y_1.

- Now, in the updated tableau, the basic variables are y_1 and w_2, and the updated basic feasible solution is $y_1 = 3$, $w_2 = 7$, all others are zero.

- We next repeat the same steps: since there is one negative number -20* in the last row, the work column is

$$\begin{bmatrix} 1/2 \\ 9/2 \\ -20 \end{bmatrix}$$

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and the ratios are

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} \div \begin{bmatrix} 1/2 \\ 9/2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14/9 \end{bmatrix} = \begin{bmatrix} 6 \\ 1.56 \end{bmatrix}$$

- Therefore, the pivot is the one with the largest ratio: $9/2^{**}$.

- w_2 is D.V. and y_2 is E.V.

- We now perform row operations to get updated simplex tableau: pivoting by first divided by $9/2$

	y_1	z_1	y_2	z_2	w_1	w_2	
y_1	1	-1	$1/2$	$-1/2$	$1/2$	0	3
w_2	0	0	1	-1	$-7/9$	$2/9$	$14/9$
	0	0	-20^*	20	60	0	360

- and then reduce all other elements in the work column to zero

	y_1	z_1	y_2	z_2	w_1	w_2	
y_1	1	-1	0	0	$1/2 + (7/9)(1/2)$	$-(2/9)(1/2)$	$3 - (14/9)(1/2)$
w_2	0	0	1	-1	$-7/9$	$2/9$	$14/9$
	0	0	0	0	$60 - 140/9$	$40/9$	$360 + 14 \times 20/9$

- simplify and replace w_2 by y_2, we obtain the updated simplex tableau:

	y_1	z_1	y_2	z_2	w_1	w_2	
y_1	1	-1	0	0	$\dfrac{8}{9}$	$-\dfrac{1}{9}$	$\dfrac{20}{9}$
y_2	0	0	1	-1	$-7/9$	$2/9$	$14/9$
	0	0	0	0	$\dfrac{400}{9}$	$40/9$	$\dfrac{3520}{9}$

- Since there is no negative number in the last row, we are done. Optimal solution:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 20/9 \\ 14/9 \end{bmatrix}, \quad \begin{bmatrix} z_1 \\ z_2 \\ w_1 \\ w_2 \end{bmatrix} = 0, \quad \text{minimal value} = -\frac{3520}{9} \approx -391.11$$

- Now, let go back to the original optimization problem:

$$\text{minimize:} \quad f(x_1, x_2) = -120x_1 - 80x_2$$
$$\text{subject to:} \quad 2x_1 + x_2 \leq 6$$
$$7x_1 + 8x_2 \leq 28.$$

- Recall that $x_1 = y_1 - z_1$, $x_2 = y_2 - z_2$. Thus, the final solution:

$$f(x_1, x_2) \text{ attains the minimal value of } -\frac{3520}{9} \approx -391.11$$
$$\text{when } x_1 = \frac{20}{9}, \quad x_2 = \frac{14}{9}.$$

Artificial LP

- The simplex method requires an initial basis.

- How to choose an initial basis?

- Brute force: arbitrarily choose m basic columns and transform the augmented matrix for the problem into canonical form. If rightmost column is positive, then we have a legitimate (initial) BFS. Otherwise, try again.

- Potentially requires $\binom{n}{m}$ tries, and tries, and is therefore not practical

- Consider the standard form LP

$$\text{minimize:} \quad z = c^T x$$
$$\text{subject to:} \quad Ax = b, \ x \geq 0,$$
$$rank(A) = m, \ A_{m \times n}, \ b \geq 0$$

- Define the associated *Artificial LP:*

$$\text{minimize:} \quad y_1 + y_2 + \cdots + y_m$$
$$\text{subject to:} \quad [A, \ I_m] \begin{bmatrix} x \\ y \end{bmatrix} = b$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \geq 0$$

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- where $y = [y_1, y_2, \cdots, y_m]^T$ is called the vector of artificial variables.

- Note that the Artificial LP has an obvious initial BFS

$$\begin{bmatrix} 0 \\ b \end{bmatrix}$$

- we can proceed with the simplex algorithm to solve the Artificial LP

- Note that for $y \geq 0$

$$y_1 + y_2 + \cdots + y_m \geq 0$$

and the strict inequality holds if one of $y_i > 0$

- So if the minimum value of the Artificial LP $= 0$, then y_i must be non-basic, i.e., $y_i = 0$

- Proposition 16.1: The original LP problem has a BFS if and only if the associated artificial problem has an optimal feasible solution with objective function value zero. In this case, the optimal solution for the artificial LP serves as a BFS for the original LP

Two-phase simplex method:

- Phase I: solve the artificial problem using simplex method

- Phase II: use the BFS resulting from phase I to initialize the simplex algorithm to solve the original LP problem.
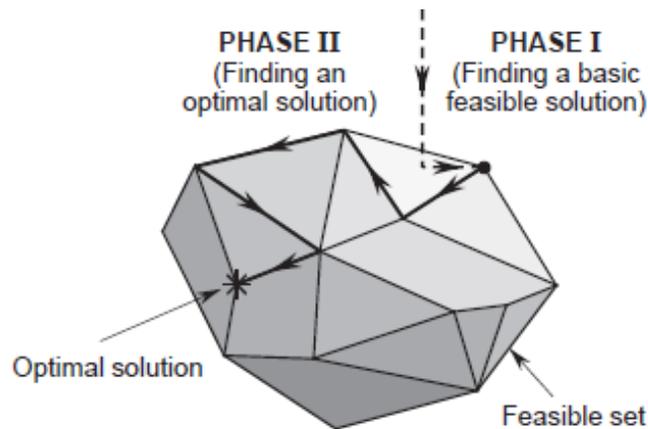
Example 16.4 Consider

$$\begin{aligned}\text{minimize:} \quad & 2x_1 + 3x_2 \\ \text{subject to:} \quad & 4x_1 + 2x_2 \geq 12 \\ & x_1 + 4x_2 \geq 6 \\ & x_1, x_2 \geq 0\end{aligned}$$

Solution:

• Standard form

$$\begin{aligned}\text{minimize:} \quad & 2x_1 + 3x_2 \\ \text{subject to:} \quad & 4x_1 + 2x_2 - x_3 = 12 \\ & x_1 + 4x_2 - x_4 = 6 \\ & x_1, ..., x_4 \geq 0\end{aligned}$$

• There is no obvious feasible solution.

• Phase I: Solve

$$\begin{aligned}\text{minimize:} \quad & x_5 + x_6 \\ \text{subject to:} \quad & 4x_1 + 2x_2 - x_3 + x_5 = 12 \\ & x_1 + 4x_2 - x_4 + x_6 = 6 \\ & x_1, ..., x_6 \geq 0\end{aligned}$$

  − $x^* = [0, 0, 0, 0, 12, 6]$ is an obvious BFS
  − simples tableau

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	4	2	−1	0	1	0	12
x_6	1	4 *	0	−1	0	1	6
	− 5	−6*	1	1	0	0	−18

  − pivoting position is $(2, 2)$: x_6 is D.V., x_2 is E.V.

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– Updated tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	$\dfrac{7}{2}^{*}$	0	-1	$\dfrac{1}{2}$	1	$-\dfrac{1}{2}$	9
x_2	$\dfrac{1}{4}$	1	0	$-\dfrac{1}{4}$	0	$\dfrac{1}{4}$	$\dfrac{3}{2}$
	$-\dfrac{7}{2}^{*}$	0	1	$-\dfrac{1}{2}$	0	$\dfrac{3}{2}$	-9

– Next, pivoting about $(1, 1) : x_5$ is D.V., x_1 is E.V.

– updating

	x_1	x_2	x_3	x_4	x_5	x_6	
x_1	1	0	$-\dfrac{2}{7}$	$\dfrac{1}{7}$	$\dfrac{2}{7}$	$-\dfrac{1}{7}$	$\dfrac{18}{7}$
x_2	0	1	$\dfrac{1}{14}$	$-\dfrac{2}{7}$	$-\dfrac{1}{14}$	$\dfrac{2}{7}$	$\dfrac{6}{7}$
	0	0	0	0	1	1	0

– Done: minimum value $= 0$. It produced a BFS $[x_1, x_2, x_3, x_4] = \left[\dfrac{18}{7}, \dfrac{6}{7}, 0, 0\right]$

• Phase II: we can obtain the simplex tableau for the original LP immediately from above:

• Deleting two artificial variable columns, then replace the last row by $c^T - c_0^T A$:

–
$$c^T = [2, 3, 0, 0], \quad c_0^T = [2, 3]$$

$$A = \begin{bmatrix} 1 & 0 & -\dfrac{2}{7} & \dfrac{1}{7} \\ 0 & 1 & \dfrac{1}{14} & -\dfrac{2}{7} \end{bmatrix}$$

$$c^T - c_0^T A = \begin{bmatrix} 0 & 0 & \dfrac{5}{14} & \dfrac{4}{7} \end{bmatrix}$$

– So the simplex tableau for the original LP :

	x_1	x_2	x_3	x_4	
x_1	1	0	$-\dfrac{2}{7}$	$\dfrac{1}{7}$	$\dfrac{18}{7}$
x_2	0	1	$\dfrac{1}{14}$	$-\dfrac{2}{7}$	$\dfrac{6}{7}$
	0	0	$\dfrac{5}{14}$	$\dfrac{4}{7}$	$-\dfrac{54}{7}$

- It is already optimal.

- The solution is $x = \begin{bmatrix} \dfrac{18}{7} & \dfrac{6}{7} & 0 & 0 \end{bmatrix}^T, f = \dfrac{54}{7}$

Homework:

1. A cereal manufacturer wishes to produce 1000 pounds of a cereal that contains exactly 10% fiber, 2% fat, and 5% sugar (by weight). The cereal is to be produced by combining four items of raw food material in appropriate proportions. These four items have certain combinations of fiber, fat, and sugar content, and are available at various prices per pound, as shown below:

Item	1	2	3	4
% fiber	3	8	16	4
% fat	6	46	9	9
% sugar	20	5	4	0
Price/lb.	2	4	1	2

The manufacturer wishes to find the amounts of each of the above items to be used to produce the cereal in the least expensive way. Formulate the problem as a linear programming problem. What can you say about the existence of a solution to this problem?

2. Solve the following linear program graphically:

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 + 5x_2 \\
\text{subject to} \quad & 0 \le x_1 \le 4 \\
& 0 \le x_2 \le 6 \\
& x_1 + x_2 \le 8
\end{aligned}
$$

3. Use the simplex method to solve the following linear program:

$$
\begin{aligned}
\text{maximize} \quad & x_1 + x_2 + 3x_3 \\
\text{subject to} \quad & x_1 + x_3 = 1 \\
& x_2 + x_3 = 2 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}
$$

4. Convert the following problem to standard form and then solve it using the simplex method:

$$
\begin{aligned}
\text{maximize} \quad & 2x_1 + x_2 \\
\text{subject to} \quad & 0 \le x_1 \le 5 \\
& 0 \le x_2 \le 7 \\
& x_1 + x_2 \le 9
\end{aligned}
$$

5. From textbook: #5.8, 5.12, 5.13, 5.16 (you may combine these four problems into one with several parts)